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ALMOST SURE COMPARISONS OF RENEWAL PROCESSES AND POISSON PROCES--ETC(U)

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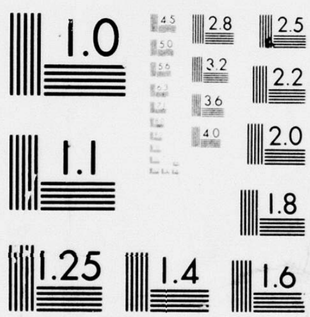

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Almost Sure Comparisons of Renewal Processes and Poisson Processes, with Application to Reliability Theory

by

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ALMOST SURE COMPARISONS OF RENEWAL PROCESSES AND POISSON PROCESSES, WITH APPLICATION TO RELIABILITY THEORY[1,2]

BY

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Abstract

If the interarrival times of a renewal process $\{ S_i, i=0,1,2... \}$ have a failure rate function which is bounded away from 0 and ∞ (infinity), then it is possible to construct (nonhomogeneous) Poisson processes $\{ T_i^0, i=0,1,2,... \}$ (superscript 0) and $\{ T_i^1, i=0,1,2,... \}$ (superscript 1) on the same probability space with $\{ S_i, i=0,1,2,... \}$ such that $\{ T_0^0, T_1^0, T_2^0,... \} \subset \{ S_0, S_1, S_2,... \} \subset \{ T_0^1, T_1^1, T_2^1,... \}$ (superscript 0, superscript 1, is a subset of) almost surely. This has applications to the reliability theory of maintained systems. An almost sure comparison is also demonstrated for certain alternating renewal processes which arose in Barlow and Proschan's (1976) investigation of maintained systems in which repairs are not instantaneous.

Key words and phrases.

Renewal process, Poisson process, stochastic inequality, almost sure inequality, reliability theory, maintained system.

1. Introduction and Summary

Inequalities play an important role in reliability theory. One example is the following: Let F be the lifetime distribution of a component or system with failure rate function $r(\cdot)$. Suppose $\inf_t r(t) = r_0 > 0$ and $\sup_t r(t) = r_1 < \infty$. If F_0 and F_1 are cdf's of exponential random variables with means r_0^{-1} and r_1^{-1} respectively, then $F_0(x) \le F(x) \le F_1(x)$ for $x \le 0$. Such inequalities are valuable in situations where the lifetime cdf F is unknown but it is possible to know or establish bounds on the failure rate function.

If a component is instantaneously replaced or repaired when it fails, the sequence of failure times forms a renewal process: Let X_i, $i=1,2,\ldots$ be a sequence of i.i.d. lifetime random variables. Let $S_0 = 0$, $S_1 = X_1$, $S_2 = X_1 + X_2$, $S_3 = X_1 + X_2 + X_3$, ... , then $\{ S_i, i=0,1,2,\ldots \}$ is a renewal process. The purpose of this note is to present bounds for the renewal process S similar to the example of the preceding paragraph, namely: Suppose that the distribution of X has failure rate function $r(\cdot)$ such that $\inf_t r(t) = r_0 > 0$ and $\sup_t r(t) = r_1 < \infty$. We show (Corollaries 1 and 2) that it is possible to construct Poisson processes $\{ T_i^0, i=0,1,2,\ldots \}$ and $\{ T_i^1, i=0,1,2,\ldots \}$ with rates r_0 and r_1 respectively, such that

$$\{ T_0^0, T_1^0, T_2^0, \ldots \} \supset \{ S_0, S_1, S_2, \ldots \} \supset \{ T_0^1, T_1^1, T_2^1, \ldots \} \qquad (1.1)$$

with probability 1 , i.e., the failure times of the respective processes are subsets of one another. (We shall actually construct nonhomogeneous Poisson processes which satisfy (1.1)).

It is possible to use this to compute bounds on expected maintenance costs, for example. This result is probably also of general interest as a contribution to the theory of Poisson processes and renewal theory and thus may be useful for establishing bounds in queueing theory or inventory theory.

It is also possible to obtain a comparison result (Theorem 3) for certain alternating renewal processes which arise when repairs or replacement are not instantaneous. Barlow and Proschan (1976) consider such processes. Theorem 3 and its proof give a slightly different and more general approach to their problem. (We assume NWU repair times; they assume DFR.)

There is a growing literature on comparison results for stochastic processes. There are two general approaches: stochastic inequalities [4,8,11,15,24] and almost sure inequalities [13,14,22]. (The list of references is by no means exhaustive.) Pledger and Proschan (1973), Ross (1974), Keilson (1974), and Barlow and Proschan (1976) have applied stochastic comparison techniques to reliability of maintained systems. The comparisons derived in this paper will be of the "almost sure" variety. These results are special for renewal and Poisson processes and are derived from first principles, not from any theories developed in the above-cited literature, for example [13,22] do not apply. The above theories tend to concentrate on stochastically monotone Markov processes [4]. We take more of a point-process point of view in this paper, focusing on sample paths and intensities rather than transition probabilities. Our approach is constructive and indicates some interesting relationships involving the failure rate functions; it may have pedagogical value.

We shall use the following relationship between stochastic and almost sure inequalities for the univariate case: Let X and Y be real-valued random variables with cdf's F and G, then X is stochastically less than Y (denoted $X \overset{st}{\leq} Y$) if $F(z) \geq G(z)$ for all z. Let U and V be real-valued random variables defined on the same probability space (Ω, F, P), then U is almost surely less than V (denoted $U \overset{as}{\leq} V$) if $P(\omega \in \Omega : U(\omega) \leq V(\omega)) = 1$. Let $X \overset{D}{=} U$ mean that the random variables X and U have the same distribution.

Lemma 1. Let X and Y be real-valued random variables. If $X \overset{st}{\leq} Y$, then there exist a probability space (Ω, F, P) and random variables U and V defined on it such that $U \overset{D}{=} X$, $V \overset{D}{=} Y$, and $U \overset{as}{\leq} V$.

Proof: Let Ω be $[0,1]$, F be the Borel sets and P be Lebesgue measure. If X and Y have cdf's F and G, let $U = F^{-1}$ and $V = G^{-1}$.

(This result can be extended to random vectors and random functions [12].)

2. Poisson Bounds on Renewal Processes

In this section the existence of (dependent) nonhomogeneous Poisson and renewal processes satisfying (1.1) on a common probability space is verified. We will use two representations of point processes on $[0,\infty)$: the counting process N and the partial sum process S. $N(t)$ = # points in $(0,t]$. $S_n = \inf\{t : N(t) \leq n\}$, $n = 1,2,\ldots$. There will be a point at 0: $S_0 = 0$; this is the zeroth point and is not counted by N. The necessary background facts about homogeneous Poisson processes, renewal processes, failure rates and other topics which will be used without reference can be found in most introductory stochastic process texts [1,6,9,19,22]. The necessary facts about nonhomogeneous Poisson processes on $[0,\infty)$ are presented below.

Definition 1 Let $r(\cdot)$ be a real-valued function on $[0,\infty)$ which is integrable over bounded intervals, then $\{ N(t), t \geq 0 \}$ is the counting process representation of a nonhomogeneous Poisson process with intensity function r if

i) $N(0) = 0$

ii) N has independent increments

iii) for $0 \leq s < t$, $N(t) - N(s)$ is a Poisson random variable with

$E(N(t) - N(s)) = \int_s^t r(u)du$.

We define $\Lambda(t) = \int_0^t r(u)du$ as the cumulative mean function of the Poisson process with intensity $r(\cdot)$.

Lemma 2. Let $\{ N(t), t \geq 0 \}$ be a nonhomogeneous Poisson process with intensity function $r(\cdot)$. Let $T_n = \inf \{ t : N(t) = n \}$, $n = 0,1,2,\ldots$. Given that $N(t) = k$, then $T_1, T_2, \ldots, T_k$ are jointly distributed as the order statistics of a sample of size k from the cdf $\Lambda(s)/\Lambda(t)$, $0 \leq s \leq t$. The density of this conditional joint distribution is

$$f_t(t_1, t_2, \ldots ,t_k \mid k) = k! \prod_{i=1}^{k} \frac{r(t_i)}{\Lambda(t)} , \quad 0 \leq t_1 \leq t_2 \leq \ldots \leq t_k \leq t .$$

Proof: Note that the conditional distribution of the interarrival time $T_{i+1} - T_i$ given $T_i = t_i$ is

$$P\{ T_{i+1} - T_i > \mid T_i = t_i \}$$

$$= P\{ N(t_i+x) - N(t_i) = 0 \mid N(t_i) = i \}$$

$$= \exp(- \int_{t_i}^{t_i+x} r(s)ds)$$

Thus a version of the density is

$$f_{T_{i+1}-T_i|T_i}(x|t_i) = r(t_i+x) \exp(- \int_{t_i}^{t_i+x} r(s)ds) .$$

The conditional density $f_t(t_1, t_2, \ldots, t_k \mid k)$ can be computed now exactly as is done in the proof of the analogous result for homogeneous Poisson processes, see [19] p.17, [1] p.67, or [9] p.126.

<u>Lemma 3</u>. Let $r(\cdot)$ be right continuous, then $\{N(t),t\geq 0\}$ is a nonhomogeneous Poisson process with intensity function $r(\cdot)$, if and only if, for $t \geq 0$,

$$\lim_{h\to 0} h^{-1} P\{ N(t+h) - N(t) = 1 \mid N(s), s \leq t \} = r(t) \tag{2.1}$$

$$\lim_{h\to 0} h^{-1} P\{ N(t+h) - N(t) \geq 2 \mid N(s), s \leq t \} = 0 \tag{2.2}$$

almost surely, and $N(0) = 0$.

<u>Proof</u>: If N is a nonhomogeneous Poisson process $N(t+h) - N(t)$ is independent of $\{ N(s), s \leq t \}$ and is Poisson with mean $\int_t^{t+h} r(u)du$. $P\{ N(t+h) - N(t) = 1 \}=\exp(- \int_t^{t+h} r(u)du) \int_t^{t+h} r(u)du = hr(t) + o(h)$. This verifies (2.1) ; (2.2) is verified similarly.

Conversely, (2.1) and (2.2) imply N is nonhomogeneous Poisson. Part (iii) of Definition 1 can be verified by solving the Kolmogorov forward differential equations as in the homogeneous case, [9] p.24, [20] p.118. Part (ii) of Definition 1 follows because the limits are independent of the entire past history, $\{ N(s), s \le t \}$: this implies the Markov property which in turn implies independent increments in this case.

Note that if (2.1) and (2.2) were weakened by conditioning over $\{ N(t) \}$ instead of $\{ N(s), s \le t \}$ then (unless the Markov property is assumed) N is not necessarily a Poisson process. Similarly part (ii) of Definition 1 cannot be weakened (to pair-wide independence, for example). Shepp (see [5]) and Szasz (1970) present counterexamples for the homogeneous case; see Renyi (1967) also.

We can now present the first comparison theorem.

<u>Theorem 1.</u> Let $\{ S_i, i=0,1,2,\dots \}$ be a renewal process with inter-arrival cdf F which has failure rate function $r(\cdot)$, i.e. $F(x) = 1 - \exp(- \int_0^x r(s)ds)$, $x \ge 0$. Let r_1 be a right-continuous function such that $\sup_{0\le s\le t} r(s) \le r_1(t) < \infty$ for $t \ge 0$. Then there exists a nonhomogeneous Poisson process $\{ T_i^1, i=0,1,2,\dots \}$ with cumulative mean function $\Lambda_1(t) = \int_0^t r_1(s)ds$ defined on the same probability space as $\{S_i, i=0,1,2,\dots\}$ such that $\{ T_0^1, T_1^1, T_2^1, \dots \} \supset \{ S_0, S_1, S_2, \dots \}$ almost surely.

<u>Proof</u>: As is frequently done, we shall not explicitly define the common probability space on which S and T^1 are defined; from the constructive definition of S from T^1 it will be clear how an appropriate probability

could be defined.

Consider $\{ T_i^1, i=0,1,2,\ldots \}$, a homogeneous Poisson process with cumulative mean function Λ_1. We shall construct a dependent version of $\{ S_i, i=0,1,2,\ldots \}$, the desired renewal process, by "thinning" T^1, i.e. certain points T_i^1 will be removed and the remaining points will constitute $\{ S_i, i=0,1,2,\ldots \}$. Define $S_0 = T_0^1 = 0$. If $T_1^1 = t_1$, remove ("thin") the point T_1^1 with probability $1-r(t_1)/r_1(t_1)$, thus $S_1 = T_1^1$ with probability $r(t_1)/r_1(t_1)$ conditional on $T_1^1 = t_1$. If T_1^1 is thinned and $T_2^1 = t_2$, remove the point T_2^1 with probability $1-r(t_2)/r_1(t_2)$. Thus, if $T_1^1 = t_1$, $T_2^1 = t_2$, $S_1 = t_2$ with probability $(1-r(t_1)/r_1(t_1))(r(t_2)/r_1(t_2))$. Continue inductively until the first "unthinned" point of T^1 is reached and define this point to be S_1. In other words, given that $T = (0, t_1, t_2, t_3, \ldots)$ define an integer-valued random variable M_1:

$$P (M_1 > m \mid T^1 = (0,t_1,t_2,\ldots)) = \prod_{i=1}^{m} \left(1 - \frac{r(t_i)}{r_1(t_i)} \right) \tag{2.3}$$

Then $S_1 = T_{M_1}^1$.

We now consider the distribution of S_1. Let $N_1(t) = \max\{ n: T_n^1 \le t \}$. Then

$$P\{ S_1 > t \} = \sum_{k=0}^{\infty} P\{ S_1 > t , N_1(t) = k \}$$

$$= \sum_{k=0}^{\infty} \int_{0 \le t_1 \le \ldots \le t_k \le t} P\{ S_1 > t \mid t_1,\ldots,t_k \} f(t_1,\ldots,t_k|k) \times dt_1\ldots dt_k P\{ N_1(t) = k \}$$

$$= \sum_{k=0}^{\infty} \int_{0 \le t_1 \le \ldots \le t_k \le t} \prod_{i=1}^{k} \left(1 - \frac{r(t_i)}{r_1(t_i)} \right) k! \prod_{i=1}^{k} \frac{r_1(t_i)}{\Lambda_1(t)} \; dt_1 \ldots dt_k \times$$

$$\exp(-\Lambda_1(t)) \frac{\Lambda_1(t)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \int_{0 \le t_1, t_2, \ldots, t_k \le t} \prod_{i=1}^{k} (r_1(t_i) - r(t_i)) \; dt_1 \ldots dt_k \frac{\exp(-\Lambda_1(t))}{k!}$$

$$= \sum_{k=0}^{\infty} \prod_{i=1}^{k} \int_0^t (r_1(s) - r(s)) ds \; \frac{\exp(-\Lambda_1(t))}{k!}$$

$$= \sum_{k=0}^{\infty} (\Lambda_1(t) - \Lambda(t))^k \frac{\exp(-\Lambda_1(t))}{k!}$$

$$= \exp(\Lambda_1(t) - \Lambda(t)) \exp(-\Lambda_1(t))$$

$$= \exp(-\Lambda(t)) = \exp(-\int_0^t r(s) ds \; .$$

The third equality follows from Lemma 2 and (2.3), the fourth by symmetry, the rest are routine. The above analysis implies that the distribution of S_1 has failure rate function $r(\cdot)$. Thus S_1 has the desired distribution.

Now we must consider the succeeding events in the renewal process S. Condition over the partition $\{ S_1 = T^1_{M_1} = s_1 \}$, $s_1 > 0$; the random index M_1 is a stopping time ([3], p.59), therefore the strong Markov property ([3], p.131) implies that the distribution of $\{ T^1_{M_1+i} - T^1_{M_1}, i=0,1,2,\ldots \}$ is independent of $\{ T^1_i, 0 \le i \le M_1 \}$ conditional on $\{T_{M_1} = s_1\}$. Given that $\{ T_{M_1} = s_1 \}$, $\{ T^1_{M_1+i} - T^1_{M_1}, i=0,1,\ldots \}$ is a nonhomogeneous Poisson process with cumulative mean function $\Lambda_{1,1}(t) = \int_{s_1}^{s_1+t} r_1(s)ds = \int_0^t r_1(s+s_1)ds$.

Note that $r_1(t+s_1) \ge \sup_{0\le s\le t+s_1} r(s) \ge \sup_{0\le s\le t} r(s)$ for all t, thus conditional on $\{ T_{M_1} = s_1 \}$, the process $\{ T^1_{M_1+i} - T^1_{M_1}, i=0,1,2,\ldots \}$ satisfies the same hypotheses which enabled us to construct $S_1 = T^1_{M_1}$. Repeating the constructive procedure gives S_2 conditional on $S_1 = s_1$. Continuing inductively allows construction of the entire renewal process $\{ S_i, i=0,1,2,\ldots \}$. This completes the proof of Theorem 1.

At this junction, some remarks are in order concerning "thinning": The usual connotation of "thinning" refers to a more random form of deleting points, for example, for $0 < p < 1$, each point is deleted with probability p independently of the process and other deletions. It is well known that a Poisson process which is "thinned" in this sense remains a Poisson process, see Renyi (1970), p.254, P.4.10. Other aspects of thinning are discussed by Råde (1972) and Jagers and Lindvall (1973).

In Theorem 1 if $\sup_{0 \le s < \infty} r(s) < \infty$ then letting $r_1 = \sup_{0 \le s < \infty} r(s)$ the hypotheses of theorem are satisfied and we have the following corollary.

<u>Corollary 1</u>. Let $\{ S_i, i=0,1,\ldots \}$ be a renewal process with interarrival distribution F which has failure rate function $r(\cdot)$. If $\sup_{0 \le t} r(t) = r_1 < \infty$ then there exists a (homogeneous) Poisson process $\{ T_i^1, i=0,1,\ldots \}$ with intensity r_1 defined on the same probability space as $\{ S_i, i=0,1,\ldots \}$ such that $\{ T_0^1, T_1^1, T_2^1, \ldots \} \supset \{ S_0, S_1, S_2, \ldots \}$ almost surely.

<u>Theorem 2</u>. Let $\{ S_i , i=0,1,2,\ldots \}$ be a renewal process with interarrival distribution F which has failure rate function $r(\cdot)$. Let r_0 be a right-continuous function such that $\inf_{s \le t} r(s) \ge r_0(t) > 0$ for $t \ge 0$. Then there exists a nonhomogeneous Poisson process $\{ T_i^0 , i=0,1,2,\ldots \}$ with cumulative mean function $\Lambda_0(t) = \int_0^t r_0(s)ds$ defined on the same probability space as $\{ S_i, i=0,1,2,\ldots \}$ such that $\{ T_0^0, T_1^0, T_2^0, \ldots \} \subset \{ S_0, S_1, S_2, \ldots \}$ almost surely.

<u>Proof</u>: Consider $\{ S_i, i=0,1,2,\ldots \}$, the renewal process with interarrival cdf F which has failure rate function r. We shall construct a dependent version of $\{ T_i^0, i=0,1,2,\ldots \}$, the desired nonhomogeneous Poisson process, by "thinning" S as follows: if $S_i = s_i$ and $S_i - S_{i-1} = x_i$, delete S_i with probability $1 - r_0(s_i)/r(x_i)$, $i=1,2,\ldots$. Let $T_0^0 = S_0$; let T_1^0 equal the smallest undeleted S_i, $i \ge 1$; let T_2^0 equal the second smallest, etc. We use Lemma 3 to show that $\{ T_i^0, i=0,1,\ldots \}$ is a nonhomogeneous Poisson

process with intensity $r_0(\cdot)$: Let $N(t) = \sup_n\{ n : S_n \le t \}$, and $N_0(t) = \sup_n\{ n : T_n^0 \le t \}$

Because T^0 is a thinning of S, the σ-fields they generate are comparable: $\sigma(\{ N_0(s), s \le t \}) \subset \sigma(\{ N(s), s \le t \})$. Therefore, in order in verify

$$\lim_{h\to 0} h^{-1} P\{ N_0(t+h) - N_0(t) = 1 \mid N_0(s), s \le t \} = r_0(t)$$

it suffices to verify

$$\lim_{h\to 0} h^{-1} P\{ N_0(t+h) - N_0(t) = 1 \mid N(s), s \le t \} = r_0(t) . \tag{2.4}$$

By the strong Markov property for renewal processes, $\{ N(s), 0 \le s \le S_{N(t)} \}$ is conditionally independent of $\{ N(s), S_{N(t)} \le s \le t \}$ given $S_{N(t)}$, furthermore since $S_{N(t)}$ is the last renewal at or before t, $\sigma(\{ N(s), S_{N(t)} \le s \le t \}) = \sigma(\{ S_{N(t)}, N(t) \})$; also, for a renewal process, $N(t+h) - N(t)$ is independent of $N(t)$ given $S_{N(t)}$. Putting all of this together gives

$$\begin{aligned} &P\{ N_0(t+h) - N_0(t) = 1 \mid N(s), s \le t \} \\ &= \sum_{k=1}^{\infty} P\{ N_0(t+h) - N_0(t) = 1 \mid N(t+h) - N(t) = k, S_{N(t)} \} \times \\ &\qquad\qquad P\{ N(t+h) - N(t) = k \mid S_{N(t)} \} . \end{aligned} \tag{2.5}$$

By renewal theory,

$$P\{ N(t+h) - N(t) = k \mid S_{N(t)} = s_* \} \le \frac{F(t+h-s_*)-F(t-s_*)}{1 - F(t-s_*)} \tag{2.6}$$

equality holds for $k = 1$, strictly inequality for $k > 2$. Dividing (2.6) by h and letting $h \downarrow 0$ yields $r(t-s_*)$ for $k = 1$ and 0 for $k > 1$. Thus in order to verify (2.4) using (2.5), it suffices to show that

$$\lim_{h\to 0} P\{ N_0(t+h) - N_0(t) = 1 \mid N(t) = 1,\ S_{N(t)} = s_* \} = \frac{r_0(t)}{r(t-s_*)} \tag{2.7}$$

But

$$P\{ N_0(t+h) - N_0(t) = 1 \mid N(t+h) - N(t) = 1,\ S_{N(t)} = s_* \}$$

$$= \int_t^{t+h} P\{S_{N(t)+1} \text{ is unthinned} \mid S_{N(t)+1} = s\}$$

$$dP\{S_{N(t)+1} = s \mid t \le S_{N(t)+1} \le t+h, S_{N(t)} = s_*\}$$

$$= \int_t^{t+h} \frac{r_0(s)}{r(s-s_*)} \frac{r(s-s_*)\ \exp(-\int_0^{s-s_*} r(u)\,du)}{\exp(-\int_0^{t-s_*} r(u)\,du) - \exp(-\int_0^{t+h-s_*} r(u)\,du)}\, ds$$

$$= \int_t^{t+h} r_0(s) \frac{\exp(-\int_{t-s_*}^{s-s_*} r(u)\,du)}{1 - \exp(-\int_{t-s_*}^{t+h-s_*} r(u)\,du)}\, ds$$

$$= \int_t^{t+h} r_0(s) \frac{\exp(-\int_{t-s_*}^{s-s_*} r(u)\,du)}{1 - 1 + r(t-s_*) + o(h)}\, ds$$

$$= \frac{1}{h} \int_t^{t+h} \frac{r_0(s)}{r(t-s_*)} ds + o(h)$$

$$= \frac{r_0(s)}{r(t-s_*)} + o(h)$$

This verifies (2.7) and consequently (2.1) of Lemma 3. Eq. (2.2) of the Lemma 3 is verified similarly. Thus T^0 is a nonhomogeneous Poisson process with intensity $r_0(\ )$.

Corollary 2. Let $\{ S_i, i=0,1,... \}$ be a renewal process with interarrival cdf F which has failure rate function $r(\ )$. If $\inf_{0 \le t < \infty} r(t) = r_0 > 0$ then there exists a (homogeneous) Poisson process $\{ T_i^0 , i=0,1,... \}$ with intensity r_0 defined on the same probability space as $\{ S_i , i=0,1,... \}$ such that $\{ T_0^0, T_1^0, T_2^0,... \} \subset \{ S_0, S_1, S_2,... \}$ almost surely.

We shall now present an example from reliability theory which indicates a possible application of the above theorems: bounds on the expected discounted cost of repair/replacement of a component renewal process. Suppose the life time of a component has distribution F with failure rate function $r(\cdot)$. Let $S_1, S_2, ...$ be successive failure times. Suppose if a component failures at time t, it costs $c(t)$ to renew it. No assumptions (except integrability) are placed on c, it can vary daily or seasonally, etc. Let d be a discount factor. The discounted cost of maintaining the system is

$$\sum_{i=1}^{\infty} \exp(-dS_i) c(S_i) .$$

If T^0 and T^1 are nonhomogeneous Poisson processes with intensity functions $r_0(\cdot)$ and $r_1(\cdot)$ respectively, as described in Theorems 1 and 2 then

$$\sum_{i=1}^{\infty} \exp(-dT_i^0)\ c(T_i^0) \le \sum_{i=1}^{\infty} \exp(-dS_i)\ c(S_i) \le \sum_{i=1}^{\infty} \exp(-dT_i^1)\ c(T_i^1)$$

It is often easier to evaluate functionals of Poisson processes than renewal processes. In the above case it is much easier to compute moments:

$$\int_0^{\infty} e^{-dt}\ c(t) r_0(t) dt \le E\left(\sum_{i=1}^{\infty} \exp(-dS_i) C(S_i)\right) \le \int_0^{\infty} e^{-dt}\ c(t) r_1(t) dt\ .$$

In this example it is plausible that $r(\cdot)$ is unknown but can be assumed to be bounded by r_0 and r_1 thus giving the above bound on expected renewal cost.

3. Comparisons of some alternating renewal processes

In this section we compare some processes which arise when failed components are not instantaneously repaired. Consider the special case of a repairable component with exponential lifetime distribution F and NWU (new worse than used) repair time distribution G. The component will alternating spend random times functioning and under repair. Let X be the performance process [2,21] :

$$X(t) = \begin{cases} 0\ , & \text{component functioning at } t \\ 1\ , & \text{component under repair at } t \end{cases}$$

If $L_1, L_2, \ldots$ and $R_1, R_2, \ldots$ are the successive lifetimes and repair times, let $T_{2i} = \sum_{j=1}^{i} (L_j + R_j)$ and $T_{2i+1} = T_{2i} + L_{i+1}$, $i = 0,1,2,\ldots$,

$$X(t) = \begin{cases} 0 & T_{2i} \le t < T_{2i+1} \\ 1 & T_{2i+1} \le t < T_{2i+2} \end{cases}$$

$i = 0,1,2,\ldots$. Let the auxiliary performance process [2] :

$$Z(t) = \begin{cases} 0 & T_{2i} \le t < T_{2i+1} \\ t - T_{2i+1} & T_{2i+1} \le t < T_{2i+2} \end{cases},$$

$i = 0,1,2,\ldots$. This process describes a system which is functioning at time 0 . Let Z_u describe a system which has been under repair at time 0 for a duration equal to u . Barlow and Proschan (1976) establish a stochastic inequality between Z_u and Z under the assumption that repair times are DFR (decreasing failure rate). We shall establish an almost-sure inequality.

Let R_0 be the initial repair period of a system under repair at time 0 . If repair commenced at $-u$, let $T^u_{-1} = -u$ and $T^u_0 = -u+R_0 > 0$. Define $T^u_i = T^u_0 + T_i$, as defined above. Then

$$Z_u(t) = \begin{cases} 0 & T^u_{2i} \le t < T^u_{2i+1} \\ t - T^u_{2i+1} & T^u_{2i+1} \le t < T^u_{2i+2} \end{cases}$$

$i = -1,0,1,\ldots$. Note that Z and Z_u are both Markov processes with the same transition law, only their initial state differs.

<u>Theorem 3</u> The processes Z and Z_u can be defined on a common probability space such that $Z(t) \le Z_u(t)$, $t \ge 0$, almost surely.

<u>Proof</u>: Let R^u_0 be the remaining repair time in a repair period which has been in progress u units of time. Let L_i be lifetime random

variables and R_i be repair time random variables, $i = 1,2,\ldots$. Let R_i^t be the remaining repair time in a repair period which has been in progress t units of time. Because R is NWU, the residual repair time R_i^t is stochastically greater than R_i, $t > 0$, $i = 1,2,\ldots$. By Lemma 1, let $I_i = [0,1]$, $\mathcal{F}_i$ be Borel sets and P_i be Lebesgue measure, define R_i , R_i^t , $t > 0$, on $(I_i, \mathcal{F}_i, P_i)$ such that $P(R_i \le R_i^t) = 1$. Define R_0^u on $(I_0, \mathcal{F}_0, P_0)$ and define L_i on similar probability spaces $(J_i, \mathcal{G}_i, Q_i)$. Let $(\Omega, \mathcal{F}, P)$ be the product space of all the above spaces; extend the domain of the above-define random variables in the obvious way. $\{ Z(t),\ t \ge 0 \}$ is automatically defined on $(\Omega, \mathcal{F}, P)$. We define $\{ Z_u(t),\ t \ge 0 \}$ as follows: On $\{ R_0^u < L_1 \}$, $Z_u(t) = u+t$ for $0 \le t < R_0^u$, and $Z_u(t) = Z(t)$ for $t \ge R_0^u$. Note that the lack of memory property of the exponential distribution guarantees that the first lifetime of the Z_u-process will have the correct distribution. On $\{ R_0^u > L_1 \}$, condition over $L_1 = \ell_1$: if $L_1 = \ell_1$ and $R_0^u \ge \ell_1$ then at time ℓ_1 (when the Z-process begins its first repair period) the Z_u-process has been under repair for a duration $u+\ell_1$, thus $R_1^{u+\ell_1}$ describes the repair status of Z_u . On the set $\{ L_1 = \ell_1 \le R_0^u ,$ $R_1^{u+\ell_1} \le R_1 + L_2 \}$ define $Z_u(t) = u+t$ for $0 \le t \le \ell_1 + R_1^{u+\ell_1}$ and $Z_u(t) = Z(t)$ for $t \ge \ell_1 + R_1^{u+\ell_1}$. By construction, the Z-process is repaired before the Z_u-process; since $R_1^{u+\ell_1} \le R_1 + L_2$, the Z-process will be functioning when the repair of the Z_u-process is completed and the residual lifetime will be exponential. This procedure of defining Z_u on $(\Omega, \mathcal{F}, P)$ can be continued inductively. The next step is to condition over $\{ L_1 = \ell_1 ,\ R_1 = r_1 ,\ L_2 = \ell_2 \}$ and $\{ R_0^u \ge \ell_1 \} \cap \{ R_1^{u+\ell_1} \ge r_1 = \ell_2 \}$.

Given these conditions, the Z_u-process has been under repair t_3+u time units at time $t_3 = \ell_1+r_1+\ell_2$, thus $R_2^{u+t_3}$ is the distribution of residual repair time. On the set $\{ L_1 = \ell_1 \geq R_0^u , R_1 = r , L_2 = \ell_2 , R_1^{u+\ell_1} \geq r_1 + \ell_2 , R_2^{u+t_3} \leq R_2+L_3 \}$ define $Z_u(t) = t+u$ for $0 \leq t \leq t_3 + R_2^{u+t_3}$ and $Z_u(t) = Z(t)$ for $t \geq t_3+R_2^{u+t_3}$. It should be clear from the construction that $Z_u(t) \geq Z(t)$ with probability 1 .

Theorem 3 can be used to establish almost sure comparisons between the operating processes of coherent systems with exponential component life times and NWU component repair times similar to the stochastic inequality established by Barlow and Proschan (1976) and then show that the time until first system failure is NBU.

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